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Interpretation of the Determinant Formulae for the Chiral Representations of the N=2 Superconformal Algebra

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ABSTRACT

We show that the N=2 determinant formulae of the Aperiodic NS algebra and the Periodic R algebra can be applied directly to incomplete Verma modules built on chiral primary states and on Ramond ground states, respectively, provided one modifies the interpretation of the zeroes in an appropriate way. That is, the zeroes of the determinat formulae account for the highest weight singular states built on chiral primaries and on Ramond ground states, but the identification of the levels and relative U(1) charges of the singular states is different than for complete Verma modules. In particular, half of the zeroes of the quadratic vanishing surfaces $f^A_{r,s} = 0$ and $f^P_{r,s} = 0$ correspond to uncharged singular states, and the other half correspond to charged singular states. We derive the spectrum of singular states built on chiral primaries, including the singular states of the Twisted Topological algebra, and the spectrum of singular states built on the Ramond ground states. We also uncover the existence of non-highest weight singular states which are not secondary of any highest weight singular state.

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1 Introduction

Various aspects concerning singular states* of the Twisted Topological N=2 Superconformal algebra have been studied in several papers during the last four years [1], [2], [3], [4], [5], [6]. In most of those papers some explicit examples of topological singular states were written down, ranging from level 2 until level 4. In addition, in [3] and [6] general formulae were given for the spectra of the uncharged singular states, that is, the spectra of U(1) charges corresponding to the topological Verma modules which contain uncharged singular states. Although the formulae fitted with the few known data, a proof or derivation was lacking.

One of the aims of this paper is precisely to derive the spectra of U(1) charges for all the four kinds of topological singular states [13]; that is, the uncharged $\mathcal{Q}_0$-closed (BRST-invariant) $|\chi\rangle^{(0)Q}$, the uncharged $\mathcal{G}_0$-closed (anti-BRST invariant) $|\chi\rangle^{(0)G}$, the charge (-1) $\mathcal{Q}_0$-closed $|\chi\rangle^{(-1)Q}$ and the charge $(+1)$ $\mathcal{G}_0$-closed $|\chi\rangle^{(1)G}$. The idea is to derive these spectra from the spectra of singular states of the untwisted Aperiodic NS algebra, *i.e.* using the corresponding determinant formula† written down for the first time by Boucher, Friedan and Kent [7] However, the determinant formula for the Aperiodic NS algebra does not apply directly to incomplete Verma modules built on chiral‡ primaries, which are precisely the only meaningful Verma modules of the Twisted Topological algebra.

A direct approach to the problem would be to compute specific "chiral" determinant formulae for such representations. The approach that we follow here is simpler, although equivalent since we actually find the zeroes of the chiral determinant formulae (they are contained in the zeroes of the determinant formula in a simple, easy to identify pattern), and we also identify the levels and relative U(1) charges of the singular states associated to them.

Although we do not present rigorous proofs, our results agree with all the known data: computation of NS singular states built on chiral primaries from level 1/2 to level 3 [13] and also the data for levels 7/2 and 4, as deduced from the computation of the topological singular states at level 4 [3]. In addition, the simplicity of our derivation strongly suggests that these results must hold at any level.

Our starting point is the ansatz that the zeroes of the determinant formula contain the zeroes of the chiral determinant formulae (one determinat formula for chiral representations and another for antichiral representations). The simplest assumption is in

*By singular state we mean null or zero-norm state. However, we will use the term singular state to denote highest weight singular states, unless otherwise indicated.

†The N=2 determinant formulae have been checked by several authors (see for example [8], [9], [10], [11] and [12]) and therefore it seems unlikely that they can contain any mistakes.

‡By chiral we mean chiral and antichiral, unless otherwise indicated.

fact that the zeroes of the chiral determinant formulae coincide with the zeroes of the determinant formula specialized to the cases $\Delta = \pm\frac{\mathsf{h}}{2}$, where Δ is the conformal weight and h the U(1) charge of the primary state on which the Verma module is built.

Then we impose the symmetries between the charged and uncharged singular states dictated by the spectral flow automorphism of the Twisted Topological algebra [5]. To be precise, this automorphism, which maps topological singular states into each other, implies after the untwisting that the charged and uncharged singular states of the Aperiodic NS algebra, built on chiral primaries, must come in pairs with a precise relation between the Verma modules to which they belong. This fact is accounted for by the determinant formula (under the simplest assumption) if half of the zeroes of the quadratic vanishing surface $f^A_{r,s} = 0$ correspond to uncharged singular states, at level $\frac{rs}{2}$, and the other half correspond to charged singular states, at level $\frac{r(s+2)-1}{2}$ (in addition to the zeroes of the vanishing plane $g^A_k = 0$). This contrasts sharply with the spectra of singular states for complete Verma modules built on non-chiral primaries [7], for which all the zeroes of the quadratic vanishing surface correspond to uncharged singular states.

The actual computation of NS singular states built on chiral primaries agrees with this analysis, showing that indeed the zeroes of the chiral determinant formulae coincide with the zeroes of the determinant formula specialized to the cases $\Delta = \pm\frac{\mathsf{h}}{2}$. The interpretation of the zeroes in terms of spectra of singular states, *i.e.* levels and relative charges, is different though. In particular, the spectrum for both charged and uncharged NS singular states built on chiral primaries is given by two-parameter expressions whereas for complete Verma modules the spectrum of charged singular states is given by a one-parameter expression *i.e.* the solutions to the vanishing plane $g^A_k = 0$.

From these two-parameter expressions we get straightforwardly the spectra of U(1) charges for the singular states of the Twisted Topological algebra, as is explained in section 3. Our expressions coincide with the expressions given in [3] and [6] accounting for the known data (until level 4).

We repeat the analysis for the Periodic R algebra for Verma modules built on the Ramond ground states. As expected, we find the same results, *i.e.* half of the zeroes of $f^P_{r,s} = 0$ are related to uncharged singular states and the other half are related to charged singular states. The reason is that the spectral flows map singular states of the Aperiodic NS algebra built on chiral primaries into singular states of the Periodic R algebra built on the Ramond ground states.

We show therefore that the zeroes of the determinant formulae of the Aperiodic NS algebra and the Periodic R algebra account for the singular states built on chiral primary states and on Ramond ground states, respectively, but the identification of the levels and relative charges of the singular states is different than for complete Verma modules.

This paper is organized as follows. In section 2 we discuss some basic facts about

the singular states of the Twisted Topological algebra and the relations between them. In section 3 we explain the direct relation between these and the singular states of the Aperiodic NS algebra. We show that the untwisting of the $\mathcal{G}_0$-closed topological singular states produces singular states of the Aperiodic NS algebra in a one-to-one way. As a consequence the charged and uncharged NS singular states built on chiral primaries must come in pairs, although in different Verma modules related to each other by the spectral flow automorphism of the Twisted Topological Algebra. We use this result in section 4, where we deduce the spectrum of U(1) charges h (and conformal weights Δ) for the NS singular states built on chiral primaries, just by analyzing the zeroes of the determinant formula and comparing the different possibilities with the actual data of singular states. The derivation of the spectrum of U(1) charges for the singular states of the Twisted Topological algebra follows straightforwardly. In section 5 we repeat the analysis for the Periodic R algebra, via the spectral flows, and we write down the spectrum of U(1) charges for the singular states built on the Ramond ground states. Section 6 is devoted to conclusions and final remarks.

Finally, in the Appendix we analyze thoroughly the NS singular states at levels 1 and $\frac{3}{2}$. We write down all the equations resulting from the highest weight conditions, showing that these equations, and therefore their solutions, are different when imposing or not chirality on the primary state on which the singular states are built. Namely, when the primary state is non-chiral all the solutions of the quadratic vanishing surface $f^A_{1,2} = 0$ correspond to level 1 uncharged singular states. When the primary state is chiral, however, half of the solutions specialized to the cases $\Delta = \pm\frac{\mathsf{h}}{2}$ correspond to level $\frac{3}{2}$ charged singular states. We also show that these charged singular states become non-highest weight singular states of a very special kind, once the chirality on the primary state is switched off, because then they are not secondary of any h.w. singular state.

2 Singular States of the Twisted Topological Algebra

The Twisted Topological algebra obtained by twisting the N=2 Superconformal algebra [14], [15], [16] reads

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\,, && [\mathcal{H}_m, \mathcal{H}_n] = \tfrac{\mathsf{c}}{3} m\delta_{m+n,0}\,,\\
&[\mathcal{L}_m, \mathcal{G}_n] = (m-n)\mathcal{G}_{m+n}\,, && [\mathcal{H}_m, \mathcal{G}_n] = \mathcal{G}_{m+n}\,,\\
&[\mathcal{L}_m, \mathcal{Q}_n] = -n\mathcal{Q}_{m+n}\,, && [\mathcal{H}_m, \mathcal{Q}_n] = -\mathcal{Q}_{m+n}\,, \qquad m,\ n \in \mathbf{Z}\,.\\
&[\mathcal{L}_m, \mathcal{H}_n] = -n\mathcal{H}_{m+n} + \tfrac{\mathsf{c}}{6}(m^2+m)\delta_{m+n,0}\,, &&\\
&\{\mathcal{G}_m, \mathcal{Q}_n\} = 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \tfrac{\mathsf{c}}{3}(m^2+m)\delta_{m+n,0}\,, &&
\end{aligned} \tag{2.1}$$

where $\mathcal{L}_m$ and $\mathcal{H}_m$ are the bosonic generators corresponding to the energy momentum

tensor (Virasoro generators) and the topological $U(1)$ current respectively, while $\mathcal{Q}_m$ and $\mathcal{G}_m$ are the fermionic generators corresponding to the BRST current and the spin-2 fermionic current respectively. The "topological central charge" c is the true central charge of the N=2 Superconformal algebra [17], [18], [9] [19] .

This algebra is satisfied by the two sets of topological generators

$$\begin{aligned}
\mathcal{L}_m^{(1)} &= L_m + \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}_m^{(1)} &= H_m\,, \\
\mathcal{G}_m^{(1)} &= G^+_{m+\frac{1}{2}}\,, \qquad \mathcal{Q}_m^{(1)} = G^-_{m-\frac{1}{2}}\,,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\mathcal{L}_m^{(2)} &= L_m - \tfrac{1}{2}(m+1)H_m\,, \\
\mathcal{H}_m^{(2)} &= -H_m\,, \\
\mathcal{G}_m^{(2)} &= G^-_{m+\frac{1}{2}}\,, \qquad \mathcal{Q}_m^{(2)} = G^+_{m-\frac{1}{2}}\,,
\end{aligned} \tag{2.3}$$

corresponding to the two possible twistings of the superconformal generators L_m, H_m, G^+_m and G^-_m. We see that G^+ and G^- play mirrored roles with respect to the definitions of $\mathcal{G}$ and $\mathcal{Q}$. In particular $(G^+_{1/2}, G^-_{-1/2})$ results in $(\mathcal{G}_0^{(1)}, \mathcal{Q}_0^{(1)})$, while $(G^-_{1/2}, G^+_{-1/2})$ gives $(\mathcal{G}_0^{(2)}, \mathcal{Q}_0^{(2)})$, so that the topological chiral primary states $|\Phi\rangle^{(1)}$ and $|\Phi\rangle^{(2)}$ (annihilated by both $\mathcal{Q}_0$ and $\mathcal{G}_0$) correspond to the antichiral and the chiral primary states of the Aperiodic NS algebra respectively. We remind the reader that the chiral and the antichiral primary states of the Aperiodic NS algebra are those primaries annihilated by $G^+_{-1/2}$ and $G^-_{-1/2}$ respectively.

As usual, only the topological primary states that are chiral will be considered. The anticommutator $\{\mathcal{G}_0, \mathcal{Q}_0\} = 2\mathcal{L}_0$ shows that these states have zero conformal weight, therefore their only quantum number is their U(1) charge h ($\mathcal{H}_0|\Phi\rangle = \mathsf{h}|\Phi\rangle$). This anticommutator also shows that a $\mathcal{Q}_0$-closed secondary state is $\mathcal{Q}_0$-exact as well, and similarly with $\mathcal{G}_0$. Therefore, the only possible physical states (BRST-closed but not BRST-exact) are the chiral primaries. Finally, this anticommutator also implies that any secondary state can be decomposed into a $\mathcal{Q}_0$-closed state and a $\mathcal{G}_0$-closed state. Therefore, we can focus on topological descendant states that are either $\mathcal{Q}_0$-closed or $\mathcal{G}_0$-closed.

A topological descendant can also be labeled by its level l (the $\mathcal{L}_0$-eigenvalue) and its U(1) charge $(\mathsf{h}+q)$ (the $\mathcal{H}_0$-eigenvalue). It is convenient to split the total U(1) charge into two pieces: the U(1) charge h of the primary state on which the descendant is built, thus labeling the corresponding Verma module $V(\mathsf{h})$, and the "relative" U(1) charge q, corresponding to the operator acting on the primary field, given by the number of $\mathcal{G}$ modes minus the number of $\mathcal{Q}$ modes in each term. $\mathcal{Q}_0$-closed and $\mathcal{G}_0$-closed topological

secondary states will be denoted as $|\chi\rangle^{(q)Q}$ and $|\chi\rangle^{(q)G}$ respectively (notice that (q) refers to the relative U(1) charge of the state).

It turns out that the topological singular states come only in four types. Namely, $|\chi\rangle^{(0)G}$, $|\chi\rangle^{(0)Q}$, $|\chi\rangle^{(1)G}$ and $|\chi\rangle^{(-1)Q}$ (the fact that these are the only kinds of topological singular states will be discussed in [13]). These four types can be mapped into each other by using $\mathcal{G}_0$, $\mathcal{Q}_0$ and the spectral flow automorphism of the Twisted Topological algebra, denoted by $\mathcal{A}$ [5]. The action of $\mathcal{G}_0$ and $\mathcal{Q}_0$ results in the following mappings

$$\begin{aligned}
\mathcal{Q}_0|\chi\rangle^{(0)G}_{\mathsf{h}} \rightarrow |\chi\rangle^{(-1)Q}_{\mathsf{h}}, \qquad & \mathcal{G}_0|\chi\rangle^{(-1)Q}_{\mathsf{h}} \rightarrow |\chi\rangle^{(0)G}_{\mathsf{h}}, \\
\mathcal{Q}_0|\chi\rangle^{(1)G}_{\mathsf{h}} \rightarrow |\chi\rangle^{(0)Q}_{\mathsf{h}}, \qquad & \mathcal{G}_0|\chi\rangle^{(0)Q}_{\mathsf{h}} \rightarrow |\chi\rangle^{(1)G}_{\mathsf{h}}.
\end{aligned}$$

Here the subindices indicate the Verma module $V(\mathsf{h})$ to which the states belong. The level of the states does not change under the action of $\mathcal{G}_0$ and $\mathcal{Q}_0$, obviously. It is important to notice that the Verma module does not change either, as indicated by the subindices. As a consequence, charged and uncharged topological singular states come always in pairs in the same Verma module. Namely, singular states of the types $|\chi\rangle^{(0)Q}$ and $|\chi\rangle^{(1)G}$ are together in the same Verma module at the same level and a similar statement holds for the singular states of the types $|\chi\rangle^{(0)G}$ and $|\chi\rangle^{(-1)Q}$.

The spectral flow automorphism of the topological algebra, on the other hand, given by [5]

$$\begin{aligned}
\mathcal{A}\,\mathcal{L}_m\,\mathcal{A} &= \mathcal{L}_m - m\mathcal{H}_m\,, \\
\mathcal{A}\,\mathcal{H}_m\,\mathcal{A} &= -\mathcal{H}_m - \tfrac{\mathsf{c}}{3}\delta_{m,0}\,, \\
\mathcal{A}\,\mathcal{Q}_m\,\mathcal{A} &= \mathcal{G}_m\,, \\
\mathcal{A}\,\mathcal{G}_m\,\mathcal{A} &= \mathcal{Q}_m\,,
\end{aligned} \tag{2.4}$$

changes the Verma module of the states as $V(\mathsf{h}) \rightarrow V(-\mathsf{h}-\frac{\mathsf{c}}{3})$. In addition, $\mathcal{A}$ reverses the relative charge as well as the BRST-invariance properties of the states, leaving the level invariant. Therefore the action of $\mathcal{A}$ results in the following mappings [5]

$$\mathcal{A}|\chi\rangle^{(0)G}_{\mathsf{h}} \rightarrow |\chi\rangle^{(0)Q}_{-\mathsf{h}-\frac{\mathsf{c}}{3}}\,, \qquad \mathcal{A}|\chi\rangle^{(-1)Q}_{\mathsf{h}} \rightarrow |\chi\rangle^{(1)G}_{-\mathsf{h}-\frac{\mathsf{c}}{3}}\,, \tag{2.5}$$

with $\mathcal{A}^{-1} = \mathcal{A}$. As a consequence, the topological singular states come in families of four, one of each kind at the same level. Two of them, one charged and one uncharged, belong to the Verma module $V(\mathsf{h})$, whereas the other pair belong to a different Verma module $V(-\mathsf{h}-\frac{\mathsf{c}}{3})$. This implies that there are not "loose" topological singular states, *i.e.* once one of them exists the other three are generated just by the action of $\mathcal{G}_0$, $\mathcal{Q}_0$ and $\mathcal{A}$, as the diagram shows.

$$
\begin{array}{ccc}
|\chi\rangle^{(0)G}_{\mathsf{h}} & \xrightarrow{\mathcal{Q}_0} & |\chi\rangle^{(-1)Q}_{\mathsf{h}} \\
\mathcal{A} \updownarrow & & \updownarrow \mathcal{A} \\
|\chi\rangle^{(0)Q}_{-\mathsf{h}-\frac{\mathsf{c}}{3}} & \xrightarrow{\mathcal{G}_0} & |\chi\rangle^{(1)G}_{-\mathsf{h}-\frac{\mathsf{c}}{3}}
\end{array}
\tag{2.6}
$$

For $\mathsf{h} = -\frac{\mathsf{c}}{6}$ the two Verma modules related by the spectral flow automorphism coincide. Therefore, if there are singular states for this value of h, they must come four by four: one of each kind at the same level.

To know if there are actually singular states for that or for any other value of h, we have to derive the spectrum of possible values of h corresponding to the topological Verma modules which contain singular states. For this purpose we first have to clarify the relation between the topological singular states and the singular states of the untwisted Aperiodic NS algebra. After doing this, we will be able to exploit the determinant formula for the Aperiodic NS algebra.

3 Untwisting the Topological Singular States

The relation between the Twisted Topological algebra and the untwisted Aperiodic NS algebra is given, obviously, by the topological twistings (2.2) and (2.3). These will be called twistings or untwistings equivalently, depending on the direction of the transformation. From the purely formal point of view, the Twisted Topological algebra (2.1) is simply a rewriting of the untwisted Aperiodic NS algebra, given by

$$
\begin{array}{lcllcl}
[L_m, L_n] & = & (m-n)L_{m+n} + \frac{\mathsf{c}}{12}(m^3-m)\delta_{m+n,0}\,, & [H_m, H_n] & = & \frac{\mathsf{c}}{3}m\delta_{m+n,0}\,, \\
[L_m, G^{\pm}_r] & = & \left(\frac{m}{2}-r\right)G^{\pm}_{m+r}\,, & [H_m, G^{\pm}_r] & = & \pm G^{\pm}_{m+r}\,, \\
[L_m, H_n] & = & -\,nH_{m+n} & & & \\
\{G^-_r, G^+_s\} & = & 2L_{r+s} - (r-s)H_{r+s} + \frac{\mathsf{c}}{3}(r^2-\frac{1}{4})\delta_{r+s,0}\,, & & &
\end{array}
\tag{3.1}
$$

where the fermionic modes take semi-integer values. The question naturally arises now whether or not the topological singular states are also a rewriting of the NS singular states. The answer is partially affirmative: only $\mathcal{G}_0$-closed topological singular states transform into NS singular states after the untwisting. Moreover, these are built on chiral primaries, when using the twist (2) (2.3), or on antichiral primaries, when using the twist (1) (2.2). The twisting of the NS singular states, on the other hand, always produces $\mathcal{G}_0$-closed topological singular states. They are built on topological chiral primaries

provided the NS singular states are built on chiral or antichiral primaries. Otherwise, the topological singular states would be built on topological primaries which are non-chiral and hence not BRST-invariant and with conformal weights different from zero. Therefore, for the purpose of twisting and untwisting we are only interested in NS singular states built on chiral or antichiral primaries.

All these statements can be verified rather easily. One only needs to investigate how the highest weight (h.w.) conditions satisfied by the singular states get modified under the twistings or untwistings given by (2.2) and (2.3). By inspecting these, it is obvious that the bosonic h.w. conditions, *i.e.* $L_{m>0}|\chi_{NS}\rangle = H_{m>0}|\chi_{NS}\rangle = 0$ on the one hand, and $\mathcal{L}_{m>0}|\chi\rangle = \mathcal{H}_{m>0}|\chi\rangle = 0$ on the other hand, are conserved under the twistings and untwistings. In other words, if the topological state $|\chi\rangle$ satisfies the topological bosonic h.w. conditions, then the corresponding untwisted state $|\chi_{NS}\rangle$ satisfies the NS bosonic h.w. conditions and vice versa. With the fermionic h.w. conditions things are not so straightforward. While the h.w. conditions $\mathcal{Q}_{m>0}|\chi\rangle = 0$ are converted into h.w. conditions of the type $G^{\pm}_{m\geq\frac{1}{2}}|\chi_{NS}\rangle = 0$ (G^{+} or G^{-} depending on the specific twist), in both twistings one of the $G^{\pm}_{1/2}$ modes is transformed into $\mathcal{G}_0$. But $G^{\pm}_{1/2}|\chi_{NS}\rangle = 0$ is nothing but a h.w. condition satisfied by all the NS singular states!

As a result, the twisting of a NS singular state (using (2.2) if it is built on an antichiral primary, or (2.3) if it is built on a chiral primary), always produces a $\mathcal{G}_0$-closed topological singular state. Conversely, only $\mathcal{G}_0$-closed topological singular states, *i.e.* those of the types $|\chi\rangle^{(0)G}$ and $|\chi\rangle^{(1)G}$, produce NS singular states under the untwistings.

Now let us analyze the transformation of the $(\mathcal{L}_0, \mathcal{H}_0)$ eigenvalues $(l, q+\mathsf{h})$ of the topological singular states $|\chi\rangle_l^{(q)G}$ into (L_0, H_0) eigenvalues $(\Delta+l, q+\mathsf{h})$ of the untwisted NS singular states $|\chi_{NS}\rangle_l^{(q)}$. In other words, given a topological singular state in the Verma module $V(\mathsf{h})$, at level l and with relative charge q, let us determine the Verma module, the level and the relative charge of the corresponding untwisted NS singular state.

Using the twist (1) (2.2) the U(1) charge does not change at all, while the level gets modified as $l - \frac{1}{2}q$. Therefore the topological singular states of the types $|\chi\rangle_l^{(0)G}$ and $|\chi\rangle_l^{(1)G}$ are transformed into NS singular states of the types $|\chi_{NS}\rangle_l^{(0)a}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ built on antichiral primaries, as is indicated by the superscript a. The Verma modules which contain the topological singular states, say $V(\mathsf{h}^{(0)})$ for the uncharged states $|\chi\rangle_l^{(0)G}$ and $V(\mathsf{h}^{(1)})$ for the charged states $|\chi\rangle_l^{(1)G}$ (with the relation $\mathsf{h}^{(1)} = -\mathsf{h}^{(0)} - \frac{\mathsf{c}}{3}$ imposed by the spectral flow automorphism of the Topological algebra), transform simply as $V_{NS}(\mathsf{h}^{(0)})$ and $V_{NS}(\mathsf{h}^{(1)})$. That is, using the twist (1), the topological Verma modules transform into NS Verma modules, but the U(1) charge of the primary states on which they are built remains unmodified. The conformal weights of the antichiral primary states are related to their U(1) charges as $\Delta = -\frac{\mathsf{h}}{2}$, as is well known.

Using the twist (2) (2.3), on the other hand, the U(1) charge reverses its sign, while

the level gets modified, again as $l - \frac{1}{2}q$. Therefore the topological singular states of the types $|\chi\rangle_l^{(0)G}$ and $|\chi\rangle_l^{(1)G}$ result in NS singular states of the types $|\chi_{NS}\rangle_l^{(0)ch}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ built on chiral primaries, as is indicated by the superscript *ch*. The corresponding Verma modules $V(\mathsf{h}^{(0)})$ and $V(\mathsf{h}^{(1)})$ are transformed into the NS Verma modules $V_{NS}(-\mathsf{h}^{(0)})$ and $V_{NS}(-\mathsf{h}^{(1)})$. The conformal weights of the chiral primary states, in turn, are related to their U(1) charges as $\Delta = \frac{\mathsf{h}}{2}$.

An interesting observation here is that the same type of topological singular state $|\chi\rangle_l^{(1)G}$ gives rise to both the charge $(+1)$ and the charge (-1) NS singular states at level $l - \frac{1}{2}$, built on antichiral primaries and chiral primaries respectively. Furthermore, since there are no topological singular states of the type $|\chi\rangle_l^{(-1)G}$, we deduce that *all* the charged NS singular states built on chiral primaries have relative U(1) charge (-1), whereas *all* the charged NS singular states built on antichiral primaries have relative U(1) charge $(+1)$. Moreover, the charge $(+1)$ and charge (-1) singular states are mirrored under the interchange $H_m \leftrightarrow -H_m$ (therefore $\mathsf{h} \leftrightarrow -\mathsf{h}$) and $G_r^+ \leftrightarrow G_r^-$.

Finally, let us stress the fact that the charged and uncharged NS singular states $|\chi_{NS}\rangle_l^{(0)a}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$, on the one hand, and $|\chi_{NS}\rangle_l^{(0)ch}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$, on the other hand, must come in pairs, although in different Verma modules, since they are just the untwistings of the topological singular states $|\chi\rangle_l^{(0)G}$ and $|\chi\rangle_l^{(1)G}$, which come in pairs inside the four-member topological families described in last section.

4 Spectrum of Topological and NS Singular States

As we have just shown, the spectrum of U(1) charges corresponding to the Verma modules which contain topological and NS singular states is the same for the singular states of the types $|\chi\rangle_l^{(0)G}$, $|\chi\rangle_l^{(-1)Q}$ and $|\chi_{NS}\rangle_l^{(0)a}$, and also the same for the singular states of the types $|\chi\rangle_l^{(0)Q}$, $|\chi\rangle_l^{(1)G}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$, where the superscript a indicates that the NS singular states are built on antichiral primaries. The first spectrum is given by all the possible values of $\mathsf{h}^{(0)}$ whereas the second one is given by all the possible values of $\mathsf{h}^{(1)}$, these values being connected to each other by the relation $\mathsf{h}^{(1)} = -\mathsf{h}^{(0)} - \frac{\mathsf{c}}{3}$.

The spectrum for the case of NS Verma modules built on chiral primaries, in turn, is the corresponding to $(-\mathsf{h}^{(0)})$, for singular states of the type $|\chi_{NS}\rangle_l^{(0)ch}$, and to $(-\mathsf{h}^{(1)})$, for singular states of the type $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$.

In principle, we cannot compute the spectra $\mathsf{h}^{(1)}$ and $\mathsf{h}^{(0)}$ of charged and uncharged singular states built on antichiral primaries, simply by imposing the relation $\Delta = -\frac{\mathsf{h}}{2}$ in the spectra given by the determinant formula of the Aperiodic NS algebra [7]. The reason is that those spectra do not apply for incomplete Verma modules constructed on chiral or antichiral primary states, *i.e.* highest weight states $|\Delta, \mathsf{h}\rangle$ which are annihilated by $G^+_{-\frac{1}{2}}$

or $G^-_{-\frac{1}{2}}$ (for which, as a result, $\Delta = \frac{\mathsf{h}}{2}$ or $\Delta = -\frac{\mathsf{h}}{2}$ respectively). Hence, it seems that one has to compute a new determinant formula specific for chiral representations. However, this is not necessary since, as we will see, the zeroes of the determinant formula contain the zeroes of the "chiral" determinant formula in a simple, easy to identify, pattern.

We start our analysis with the ansatz that the set of zeroes of the chiral determinant formula is included in the set of zeroes of the general determinant formula for the particular cases $\Delta = \frac{\mathsf{h}}{2}$ (or $\Delta = -\frac{\mathsf{h}}{2}$), with possibly a different interpretation in terms of h.w. singular states. Although this ansatz seems rather intuitive, the proof is not straightforward at all. The reason is that the equations obtained by imposing the h.w. conditions on a given secondary state depend on whether the primary state is chiral or not. As a consequence, the h.w. conditions give different h.w. singular states for chiral representations than for non-chiral representations (see the Appendix).

Our strategy will be now to analyze the zeroes of the determinant formula taking into account that the spectral flow automorphism of the topological algebra predicts an equal number of charged and uncharged singular states (built on chiral or antichiral primaries) with a precise relation between their corresponding Verma modules. Namely, for singular states on antichiral modules the relation is $\mathsf{h}^{(1)} = -\mathsf{h}^{(0)} - \frac{\mathsf{c}}{3}$, while for singular states on chiral modules it is $\mathsf{h}^{(1)} = -\mathsf{h}^{(0)} + \frac{\mathsf{c}}{3}$ (in this last expression $\mathsf{h}^{(1)}$ and $\mathsf{h}^{(0)}$ denote the spectra for the chiral modules, instead of $(-\mathsf{h}^{(1)})$ and $(-\mathsf{h}^{(0)})$ in our previous notation).

Let us concentrate on the antichiral modules, for convenience. The zeroes of the determinant formula of the Aperiodic NS algebra [7] are given by the solutions of the quadratic vanishing surface $f^A_{rs} = 0$, with

$$f^A_{r,s} = 2\left(\frac{\mathsf{c}-3}{3}\right)\Delta - \mathsf{h}^2 - \frac{1}{4}\left(\frac{\mathsf{c}-3}{3}\right)^2 + \frac{1}{4}\left(\left(\frac{\mathsf{c}-3}{3}\right)r + s\right)^2 \qquad r \in \mathbf{Z}^+\,,\ s \in 2\mathbf{Z}^+ \quad (4.1)$$

and the solutions of the vanishing plane $g^A_k = 0$, with

$$g^A_k = 2\Delta - 2k\mathsf{h} + \left(\frac{\mathsf{c}-3}{3}\right)(k^2 - \frac{1}{4}) \qquad k \in \mathbf{Z} + \frac{1}{2} \quad (4.2)$$

Solving for $f^A_{r,s} = 0$, with $\Delta = -\frac{\mathsf{h}}{2}$, one finds two two-parameter solutions for h (since $f^A_{r,s} = 0$ becomes a quadratic equation for h). These solutions are

$$\mathsf{h}_{r,s} = -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)(r+1) + s\right) \quad (4.3)$$

and

$$\hat{\mathsf{h}}_{r,s} = \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)(r-1) + s\right)\,. \quad (4.4)$$

Solving for $g_k^A = 0$, with $\Delta = -\frac{\mathsf{h}}{2}$, one finds the one-parameter solution

$$\mathsf{h}_k = \left(\frac{\mathsf{c}-3}{6}\right)(k - \frac{1}{2})\,, \tag{4.5}$$

except for $k = -\frac{1}{2}$ where g_k^A is identically zero.

There are two different situations to be considered now, depending on whether the number of zeroes of the chiral determinant formula is equal or smaller than the number of zeroes given by the solutions (4.3), (4.4) and (4.5).

Let us start with the first situation. The problem at hand is therefore to distribute all these solutions into two sets, say $H^{(0)}$ and $H^{(1)}$, such that for any given solution $\mathsf{h}^{(0)}$ in the set $H^{(0)}$ there exists one solution $\mathsf{h}^{(1)}$ in the set $H^{(1)}$, satisfying $\mathsf{h}^{(1)} = -\mathsf{h}^{(0)} - \frac{\mathsf{c}}{3}$, and vice versa. For this purpose it is helpful to write down the set of expressions corresponding to $(-\mathsf{h}_{r,s} - \frac{\mathsf{c}}{3})$, $(-\hat{\mathsf{h}}_{r,s} - \frac{\mathsf{c}}{3})$ and $(-\mathsf{h}_k - \frac{\mathsf{c}}{3})$. These are given by

$$-\mathsf{h}_{r,s} - \frac{\mathsf{c}}{3} \;=\; \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)(r-1) + s - 2\right)\,, \tag{4.6}$$

$$-\hat{\mathsf{h}}_{r,s} - \frac{\mathsf{c}}{3} \;= -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)(r+1) + s + 2\right) \tag{4.7}$$

and

$$-\mathsf{h}_k - \frac{\mathsf{c}}{3} \;= -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)(k + \frac{3}{2}) + 2\right)\,. \tag{4.8}$$

Comparing these expressions with the set of solutions $\mathsf{h}_{r,s}$, $\hat{\mathsf{h}}_{r,s}$ and h_k , given by (4.3), (4.4) and (4.5), one finds straightforwardly

$$\mathsf{h}_{r,2} = -\mathsf{h}_{r-\frac{1}{2}} - \frac{\mathsf{c}}{3}\,, \qquad \mathsf{h}_{r,s>2} = -\hat{\mathsf{h}}_{r,(s-2)} - \frac{\mathsf{c}}{3}\,. \tag{4.9}$$

Therefore one solution to the problem is that the spectrum of the uncharged singular states is given by $\mathsf{h}^{(0)}_{r,s} = \mathsf{h}_{r,s}$, with the level of the state $l = \frac{rs}{2}$, whereas the spectrum of the charged singular states, at level $l - \frac{1}{2} = \frac{rs-1}{2}$, is given by $\mathsf{h}^{(1)}_{r,s} = -\mathsf{h}_{r,s} - \frac{\mathsf{c}}{3}$, that is

$$\mathsf{h}^{(1)}_{r,s} = \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)(r-1) + s - 2\right)\,, \tag{4.10}$$

which contains the two series h_k and $\hat{\mathsf{h}}_{r,s}$. Namely, for $s = 2$ $\mathsf{h}^{(1)}_{l,2} = \mathsf{h}_{l-\frac{1}{2}}$, where $l - \frac{1}{2}$ is the level of the charged singular state, while for $s > 2$ $\mathsf{h}^{(1)}_{r,s>2} = \hat{\mathsf{h}}_{r,(s-2)}$, with the level $l - \frac{1}{2}$ given by $l - \frac{1}{2} = \frac{rs-1}{2}$.

We see therefore that in this solution half of the zeroes of the quadratic vanishing surface $f^A_{r,s} = 0$ correspond to uncharged singular states at level $l = \frac{rs}{2}$, and the other half correspond to charged singular states at level $l - \frac{1}{2} = \frac{r(s+2)-1}{2}$. The zeroes of the vanishing plane $g^A_k = 0$, in turn, correspond to charge $(+1)$ singular states at level k, built on antichiral primaries, for $k > 0$, and to charge (-1) singular states at level $(-k)$, built on chiral primaries, for $k < 0$ (the reader can verify this last statement by repeating our analysis imposing chirality rather than antichirality on the primary states).

The second situation, in which the number of zeroes of the chiral determinant formula is smaller than the number of zeroes of the general determinant formula has, in principle, more solutions than the previous one. The simplest one is to set $\mathsf{h}^{(0)} = \mathsf{h}_{l,2}$, $\mathsf{h}^{(1)} = -\mathsf{h}_{l,2} - \frac{\mathsf{c}}{3} = \mathsf{h}_{l-\frac{1}{2}}$. However, before searching for more intricate possibilities let us have a look at the data for $\mathsf{h}^{(0)}$ and $\mathsf{h}^{(1)}$ given by the singular states themselves. We have computed until level 3, just by imposing the h.w. conditions, all the topological singular states and all the NS singular states built on chiral and antichiral primaries (some of these singular states were already published, of course). The explicit expressions for all these singular states will be given in [13], although in the Appendix we also write down and analyze the NS singular states $|\chi_{NS}\rangle^{(0)}_1$, $|\chi_{NS}\rangle^{(1)a}_{\frac{3}{2}}$ and $|\chi_{NS}\rangle^{(-1)ch}_{\frac{3}{2}}$. Here we need only the values of $\mathsf{h}^{(0)}$ and $\mathsf{h}^{(1)}$. These are the following:

- For $|\chi\rangle^{(0)G}_1$, $|\chi\rangle^{(-1)Q}_1$ and $|\chi_{NS}\rangle^{(0)a}_1$ $\mathsf{h}^{(0)} = -\frac{\mathsf{c}}{3}$

- For $|\chi\rangle^{(0)Q}_1$, $|\chi\rangle^{(1)G}_1$ and $|\chi_{NS}\rangle^{(1)a}_{\frac{1}{2}}$ $\mathsf{h}^{(1)} = 0$

- For $|\chi\rangle^{(0)G}_2$, $|\chi\rangle^{(-1)Q}_2$ and $|\chi_{NS}\rangle^{(0)a}_2$ $\mathsf{h}^{(0)} = \frac{1-\mathsf{c}}{2}$, $-\frac{\mathsf{c}+3}{3}$

- For $|\chi\rangle^{(0)Q}_2$, $|\chi\rangle^{(1)G}_2$ and $|\chi_{NS}\rangle^{(1)a}_{\frac{3}{2}}$ $\mathsf{h}^{(1)} = \frac{\mathsf{c}-3}{6}$, 1

- For $|\chi\rangle^{(0)G}_3$, $|\chi\rangle^{(-1)Q}_3$ and $|\chi_{NS}\rangle^{(0)a}_3$ $\mathsf{h}^{(0)} = \frac{3-2\mathsf{c}}{3}$, $-\frac{\mathsf{c}+6}{3}$

- For $|\chi\rangle^{(0)Q}_3$, $|\chi\rangle^{(1)G}_3$ and $|\chi_{NS}\rangle^{(1)a}_{\frac{5}{2}}$ $\mathsf{h}^{(1)} = \frac{\mathsf{c}-3}{3}$, 2 .

In addition, the BRST-invariant uncharged topological singular state at level 4 $|\chi\rangle^{(0)Q}_4$ has been computed in [3] with the result $\mathsf{h}^{(1)} = \frac{\mathsf{c}-3}{2}$, $\frac{\mathsf{c}+3}{6}$, 3. These values also correspond to the singular states $|\chi\rangle^{(1)G}_4$ and $|\chi_{NS}\rangle^{(1)a}_{\frac{7}{2}}$, as we have deduced in section 3.

By comparing these results with the zeroes of the determinant formula, given by expressions (4.3), (4.4) and (4.5), we notice that the values we have found for $\mathsf{h}^{(0)}$ fit exactly in the expression $\mathsf{h}_{r,s}$ (4.3), that is, the upper solution for the quadratic vanishing surface, but not in the lower solution $\hat{\mathsf{h}}_{r,s}$. The values we have found for $\mathsf{h}^{(1)}$ follow exactly

the prediction of the spectral flow automorphism for each case, *i.e.* $\mathsf{h}^{(1)} = -\mathsf{h}^{(0)} - \frac{\mathsf{c}}{3}$. Therefore we can set $\mathsf{h}^{(0)}_{r,s} = \mathsf{h}_{r,s}$ and $\mathsf{h}^{(1)}_{r,s} = -\mathsf{h}_{r,s} - \frac{\mathsf{c}}{3}$. Hence, the actual spectra of singular states follow exactly the pattern we have found under the ansatz that the zeroes of the chiral determinant formula coincide with the zeroes of the determinant formula (for $\Delta = -\frac{\mathsf{h}}{2}$ in the antichiral case). That is, $\mathsf{h}^{(0)}_{r,s}$ is given by (4.3) and $\mathsf{h}^{(1)}_{r,s}$ is given by (4.10).

For the case of NS singular states built on chiral primaries one finds the same values for the U(1) charges as for the antichiral case, but with the sign reversed, as expected. Therefore the spectra of U(1) charges for the NS singular states of the types $|\chi_{NS}\rangle^{(0)ch}_{l}$ and $|\chi_{NS}\rangle^{(-1)ch}_{l-\frac{1}{2}}$ are given by $(-\mathsf{h}^{(0)}_{r,s})$ and $(-\mathsf{h}^{(1)}_{r,s})$ respectively.

Let us summarize our results for the spectra of U(1) charges and conformal weights corresponding to the Verma modules which contain the different kinds of singular states.

*) The spectrum of U(1) charges corresponding to topological singular states of the types $|\chi\rangle^{(0)G}_{l}$ and $|\chi\rangle^{(-1)Q}_{l}$, and uncharged NS singular states built on antichiral primaries $|\chi_{NS}\rangle^{(0)a}_{l}$ is given by $\mathsf{h}^{(0)}_{r,s}$, eq.(4.3), where the level of the singular states is given by $l = \frac{rs}{2}$. The spectrum of conformal weights for the NS singular states $|\chi_{NS}\rangle^{(0)a}_{l}$ is given therefore by $\Delta^{(0)}_{r,s} = -\frac{1}{2}\mathsf{h}^{(0)}_{r,s}$.

*) The spectra of U(1) charges and conformal weights corresponding to uncharged NS singular states built on chiral primaries $|\chi_{NS}\rangle^{(0)ch}_{l}$ are given by $(-\mathsf{h}^{(0)}_{r,s})$ and $\Delta^{(0)}_{r,s} = \frac{1}{2}(-\mathsf{h}^{(0)}_{r,s})$. Therefore the spectrum of conformal weights corresponding to uncharged NS singular states is the same for those built on chiral primaries than for those built on antichiral primaries, although the spectrum of U(1) charges is reversed in sign.

*) The spectrum of U(1) charges corresponding to topological singular states of the types $|\chi\rangle^{(0)Q}_{l}$ and $|\chi\rangle^{(1)G}_{l}$, and charged NS singular states built on antichiral primaries $|\chi_{NS}\rangle^{(1)a}_{l-\frac{1}{2}}$ is given by $\mathsf{h}^{(1)}_{r,s} = -\mathsf{h}^{(0)}_{r,s} - \frac{\mathsf{c}}{3}$, resulting in expression (4.10), where $l = \frac{rs}{2}$. The spectrum of conformal weights for the NS singular states $|\chi_{NS}\rangle^{(1)a}_{l-\frac{1}{2}}$ is given therefore by $\Delta^{(1)}_{r,s} = -\frac{1}{2}\mathsf{h}^{(1)}_{r,s}$.

*) The spectra of U(1) charges and conformal weights corresponding to charged NS singular states built on chiral primaries $|\chi_{NS}\rangle^{(-1)ch}_{l-\frac{1}{2}}$ are given by $(-\mathsf{h}^{(1)}_{r,s})$ and $\Delta^{(1)}_{r,s} = \frac{1}{2}(-\mathsf{h}^{(1)}_{r,s})$. Therefore the spectrum of conformal weights corresponding to charged NS singular states is the same for those built on chiral primaries than for those built on antichiral primaries, although the spectrum of U(1) charges, as well as the relative charge, is reversed in sign.

Comparing these spectra with the BFK spectra for non-chiral representations, one finds the remarkable fact that half of the uncharged singular states with levels $l = \frac{rs}{2}$, in the non-chiral Verma modules, have been replaced by charged singular states with levels $l - \frac{1}{2} = \frac{r(s+2)-1}{2}$, in the chiral modules. Namely those singular states with h given by $\pm\hat{\mathsf{h}}_{rs}$ in (4.4). In the Appendix we analyze this effect for the particular case of the uncharged singular states $|\chi_{NS}\rangle_1^{(0)}$ and the charged singular states $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$ and $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$. We write down the h.w. equations, with their solutions, for the primary states being non-chiral, chiral and antichiral.

An interesting question now is which kind of states become the uncharged singular states with $\mathsf{h} = \pm\hat{\mathsf{h}}_{rs}$ once we switch on chirality on the primary states. An equally interesting question is which kind of states are the charged singular states with $\mathsf{h} = \pm\hat{\mathsf{h}}_{rs}$ once we switch off chirality on the primary states. We do not know the answer in general. In the Appendix we show that the uncharged singular states $|\chi_{NS}\rangle_1^{(0)}$, for $\Delta = \mp\frac{\mathsf{h}}{2}$, $\mathsf{h} = \pm\hat{\mathsf{h}}_{1,2} = \pm 1$ vanish once we switch on antichirality and chirality on the primary states, respectively. The charged singular states $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ with $\mathsf{h} = 1$ and $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$ with $\mathsf{h} = -1$, on the other hand, become non-highest weight singular states after we switch off antichirality and chirality respectively on the primary states. These singular states are very special since, in spite of being non-highest weight states, they are not secondary states of any h.w. singular states either (they can descend down to a h.w. singular state but not the other way around).

5 Spectrum of R Singular States

The singular states of the Aperiodic NS algebra transform into singular states of the Periodic R algebra under the action of the spectral flows, and vice versa [20], [9], [19], [21]. In particular, the NS singular states built on chiral or antichiral primaries transform into R singular states built on the Ramond ground states. As a consequence, we can write down easily the spectrum of U(1) charges for the R singular states built on the Ramond ground states simply by applying the spectral flow transformations to the spectra (4.3) and (4.10) found in last section. Before doing this let us say a few words about the Periodic R algebra.

The Periodic N=2 Superconformal algebra is given by (3.1), where the fermionic generators $G_r^{\pm}$ are integer moded. The zero modes of the fermionic generators characterize the states as being G_0^+-closed or G_0^--closed, as the anticommutator $\{G_0^+, G_0^-\} = 2L_0 - \frac{\mathsf{c}}{12}$ shows. The Ramond ground states are annihilated by both G_0^+ and G_0^-, therefore $\Delta = \frac{\mathsf{c}}{24}$ for them, as a result.

In order to simplify the analysis that follows it is very convenient to define the U(1)

charge for the states of the Periodic R algebra in the same way as for the states of the Aperiodic NS algebra. Namely, the U(1) charge of the states will be denoted by h, instead of $\mathsf{h}\pm\frac{1}{2}$, whereas the relative charge q of a secondary state will be defined as the difference between the H_0-eigenvalue of the state and the H_0-eigenvalue of the primary on which it is built. Therefore, the relative charges of the R states are defined to be integer, in contrast with the usual definition.

The spectral flow [20], [19], is given by the one-parameter family of transformations

$$\begin{aligned}
\mathcal{U}_\theta\, L_m\, \mathcal{U}_\theta^{-1} &= L_m + \theta H_m + \tfrac{\mathsf{c}}{6}\theta^2\delta_{m,0}\,,\\
\mathcal{U}_\theta\, H_m\, \mathcal{U}_\theta^{-1} &= H_m + \tfrac{\mathsf{c}}{3}\theta\delta_{m,0}\,,\\
\mathcal{U}_\theta\, G_r^+\, \mathcal{U}_\theta^{-1} &= G_{r+\theta}^+\,,\\
\mathcal{U}_\theta\, G_r^-\, \mathcal{U}_\theta^{-1} &= G_{r-\theta}^-\,,
\end{aligned} \tag{5.1}$$

satisfying $\mathcal{U}_\theta^{-1} = \mathcal{U}_{(-\theta)}$ and giving rise to isomorphic algebras. If we denote by (Δ, h) the (L_0, H_0) eigenvalues of any given state, then the eigenvalues of the transformed state $\mathcal{U}_\theta|\chi\rangle$ are $(\Delta - \theta\mathsf{h} + \frac{c}{6}\theta^2, \mathsf{h} - \frac{c}{3}\theta)$. If the state $|\chi\rangle$ is a level-l secondary state with relative charge q and eigenvalues $(\Delta + l, \mathsf{h} + q)$ (where now (Δ, h) denote the eigenvalues of the primary on which the secondary is built), then one gets straightforwardly that the level of the transformed state $\mathcal{U}_\theta|\chi\rangle$ changes to $l - \theta q$, while the relative charge remains equal.

There is another spectral flow [21], which is the untwisting of the topological algebra automorphism, given by

$$\begin{aligned}
\mathcal{A}_\theta\, L_m\, \mathcal{A}_\theta &= L_m + \theta H_m + \tfrac{\mathsf{c}}{6}\theta^2\delta_{m,0}\,,\\
\mathcal{A}_\theta\, H_m\, \mathcal{A}_\theta &= -H_m - \tfrac{\mathsf{c}}{3}\theta\delta_{m,0}\,,\\
\mathcal{A}_\theta\, G_r^+\, \mathcal{A}_\theta &= G_{r-\theta}^-\,,\\
\mathcal{A}_\theta\, G_r^-\, \mathcal{A}_\theta &= G_{r+\theta}^+\,.
\end{aligned} \tag{5.2}$$

with $\mathcal{A}_\theta^{-1} = \mathcal{A}_\theta$. The (L_0, H_0) eigenvalues of the transformed states $\mathcal{A}_\theta|\chi\rangle$ are now $(\Delta + \theta\mathsf{h} + \frac{c}{6}\theta^2, -\mathsf{h} - \frac{c}{3}\theta)$ (that is, they differ from the previous case by the interchange $\mathsf{h} \to -\mathsf{h}$). From this one easily deduces that, under the spectral flow (5.2), the level l of any descendant will change to $l + \theta q$ while the relative charge q reverse its sign.

For half-integer values of θ the two spectral flows interpolate between the Aperiodic NS algebra and the Periodic R algebra. In particular, for $\theta = \frac{1}{2}$ the primaries of the NS algebra (including singular states) become primaries of the R algebra with chirality $(-)$ (*i.e.* annihilated by G_0^-), while for $\theta = -\frac{1}{2}$ the primaries of the NS algebra become primaries of the R algebra with chirality $(+)$ (*i.e.* annihilated by G_0^+). In addition, $\mathcal{U}_{1/2}$ and $\mathcal{A}_{-1/2}$ map the chiral primaries of the NS algebra (*i.e.* annihilated by $G_{-1/2}^+$) into the set of Ramond ground states, whereas $\mathcal{U}_{-1/2}$ and $\mathcal{A}_{1/2}$ map the antichiral primaries (*i.e.* annihilated by $G_{-1/2}^-$) into the set of Ramond ground states. As a result, the spectral

flows (5.1) and (5.2) transform the NS singular states built on chiral and antichiral primaries into R singular states built on the Ramond ground states, as we said before. We will denote the R states as $|\chi_R\rangle_l^{(q)+}$ and $|\chi_R\rangle_l^{(q)-}$, where, in addition to the level and the relative charge, we indicate that the state is annihilated by G_0^+ or G_0^-.

As we showed in section 3, the NS singular states built on chiral primaries are only of two types, $|\chi_{NS}\rangle_l^{(0)ch}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$, and similarly, the NS singular states built on antichiral primaries come only in two types $|\chi_{NS}\rangle_l^{(0)a}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$. In fact, $|\chi_{NS}\rangle_l^{(0)ch}$ and $|\chi_{NS}\rangle_l^{(0)a}$, on the one hand, and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$, on the other hand, are mirrored under the interchange $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$, because they are the two possible untwistings of the $\mathcal{G}_0$-closed topological singular states $|\chi\rangle_l^{(0)G}$ and $|\chi\rangle_l^{(1)G}$ respectively.

From the discussion above one deduces easily that the NS singular states $|\chi_{NS}\rangle_l^{(0)a}$, $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$, $|\chi_{NS}\rangle_l^{(0)ch}$ and $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ are transformed into R singular states in the following way:

$$\mathcal{A}_{1/2}\ |\chi_{NS}\rangle_l^{(0)a} = \mathcal{U}_{1/2}\ |\chi_{NS}\rangle_l^{(0)ch} = |\chi_R\rangle_l^{(0)-} \tag{5.3}$$

$$\mathcal{A}_{1/2}\ |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a} = \mathcal{U}_{1/2}\ |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch} = |\chi_R\rangle_l^{(-1)-} \tag{5.4}$$

$$\mathcal{U}_{-1/2}\ |\chi_{NS}\rangle_l^{(0)a} = \mathcal{A}_{-1/2}\ |\chi_{NS}\rangle_l^{(0)ch} = |\chi_R\rangle_l^{(0)+} \tag{5.5}$$

$$\mathcal{U}_{-1/2}\ |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a} = \mathcal{A}_{-1/2}\ |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch} = |\chi_R\rangle_l^{(1)+} \tag{5.6}$$

where the NS Verma modules $V_{NS}(\mathsf{h})$ are transformed, in turn, as

$$\begin{aligned} \mathcal{U}_{1/2}\ V_{NS}(\mathsf{h}) &\to V_R(\mathsf{h} - \tfrac{\mathsf{c}}{6})\ , \quad & \mathcal{U}_{-1/2}\ V_{NS}(\mathsf{h}) &\to V_R(\mathsf{h} + \tfrac{\mathsf{c}}{6})\ , \\ \mathcal{A}_{1/2}\ V_{NS}(\mathsf{h}) &\to V_R(-\mathsf{h} - \tfrac{\mathsf{c}}{6})\ , \quad & \mathcal{A}_{-1/2}\ V_{NS}(\mathsf{h}) &\to V_R(-\mathsf{h} + \tfrac{\mathsf{c}}{6})\ . \end{aligned}$$

Observe that the charged singular states built on the Ramond ground states come only in two types: $|\chi_R\rangle_l^{(1)+}$ and $|\chi_R\rangle_l^{(-1)-}$, *i.e.* the sign of the relative charge is equal to the sign of the chirality.

Hence, whereas the spectrum of conformal weights for all these R singular states is just given by $\Delta = \frac{\mathsf{c}}{24}$, the spectrum of U(1) charges is given by: $\mathsf{h}_{r,s}^{(0)+} = \mathsf{h}_{r,s}^{(0)} + \frac{\mathsf{c}}{6}$ for singular states of type $|\chi_R\rangle_l^{(0)+}$, $\mathsf{h}_{r,s}^{(1)+} = \mathsf{h}_{r,s}^{(1)} + \frac{\mathsf{c}}{6}$ for singular states of type $|\chi_R\rangle_l^{(1)+}$, $\mathsf{h}_{r,s}^{(0)-} = -(\mathsf{h}_{r,s}^{(0)} + \frac{\mathsf{c}}{6})$ for singular states of type $|\chi_R\rangle_l^{(0)-}$ and $\mathsf{h}_{r,s}^{(-1)-} = -(\mathsf{h}_{r,s}^{(1)} + \frac{\mathsf{c}}{6})$ for singular states of type $|\chi_R\rangle_l^{(-1)-}$.

Using the expressions for $\mathsf{h}^{(0)}_{r,s}$ and $\mathsf{h}^{(1)}_{r,s}$ given by (4.3) and (4.10) we obtain finally the spectra of U(1) charges for the R singular states built on the Ramond ground states:

$$\mathsf{h}^{(0)+}_{r,s} = -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right) r + s - 1\right) \quad \text{for} \qquad |\chi_R\rangle_l^{(0)+} \ , \tag{5.7}$$

$$\mathsf{h}^{(0)-}_{r,s} = \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right) r + s - 1\right) \quad \text{for} \qquad |\chi_R\rangle_l^{(0)-} \ , \tag{5.8}$$

$$\mathsf{h}^{(1)+}_{r,s} = \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right) r + s - 1\right) \quad \text{for} \qquad |\chi_R\rangle_l^{(1)+} \tag{5.9}$$

and

$$\mathsf{h}^{(-1)-}_{r,s} = -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right) r + s - 1\right) \quad \text{for} \qquad |\chi_R\rangle_l^{(-1)-} \ . \tag{5.10}$$

We see that $|\chi_R\rangle_l^{(0)+}$ and $|\chi_R\rangle_l^{(-1)-}$ are together in the same Verma module $V_R(\mathsf{h})$ and at the same level, and so are $|\chi_R\rangle_l^{(0)-}$ and $|\chi_R\rangle_l^{(1)+}$, which belong to the Verma module $V_R(-\mathsf{h})$. Therefore, the singular states built on the Ramond ground states come always in sets of two pairs at the same level. Every pair consists of one charged and one uncharged state, with opposite chiralities, one pair belonging to the Verma module $V_R(\mathsf{h})$ and the other to the Verma module $V_R(-\mathsf{h})$. This is easy to see also taking into account that the action of G_0^+ or G_0^- on any singular state built on the Ramond ground states produces another singular state with different relative charge and different chirality but with the same level and sitting in the same Verma module (since both G_0^+ and G_0^- annihilate the Ramond ground states).

This resembles very much the family structure for the Twisted Topological algebra that we analyzed in section 2. However, there is a drastic difference here because in the latter case the four members of the topological family, *i.e.* $|\chi\rangle_l^{(0)G}$, $|\chi\rangle_l^{(0)Q}$, $|\chi\rangle_l^{(1)G}$ and $|\chi\rangle_l^{(-1)Q}$, are completely different from each other, while in this case the four members are two by two mirror symmetric under the interchange $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$.

Now let us compare the spectra we have found, eqns. (5.7), (5.8), (5.9) and (5.10), with the zeroes of the determinant formula for the Periodic R algebra for $\Delta = \frac{\mathsf{c}}{24}$. Let us remind that our definition of U(1) charge is different from the definition given in [7]. For example, our uncharged states are called charge $(-\frac{1}{2}sgn(0))$ states there.

The zeroes of the determinant formula for the Periodic R algebra are given by the vanishing quadratic surface $f^P_{r,s} = 0$ and the vanishing plane $g^P_k = 0$, where

$$f^P_{r,s} = 2\left(\frac{\mathsf{c}-3}{3}\right)(\Delta - \frac{\mathsf{c}}{24}) - \mathsf{h}^2 + \frac{1}{4}\left(\left(\frac{\mathsf{c}-3}{3}\right) r + s\right)^2 \qquad r \in \mathbf{Z}^+ \, , \ s \in 2\mathbf{Z}^+ \tag{5.11}$$

and

$$g_k^P = 2\Delta - 2k\mathsf{h} + \left(\frac{\mathsf{c}-3}{3}\right)(k^2 - \frac{1}{4}) - \frac{1}{4} \qquad k \in \mathbf{Z} + \frac{1}{2}\,. \tag{5.12}$$

For $\Delta = \frac{\mathsf{c}}{24}$ they result in the following solutions

$$\mathsf{h}_{r,s}^{BFK} = \pm\,\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)r + s\right) \tag{5.13}$$

and

$$\mathsf{h}_k^{BFK} = \frac{1}{2}\left(\frac{\mathsf{c}-3}{3}\right)k \tag{5.14}$$

where the superscript indicates that these are U(1) charges in the notation of BFK. Therefore we have to add $(\pm\frac{1}{2})$ to these expressions to translate them into our notation. Namely,

$$\mathsf{h}_{r,s}^{BFK} + \frac{1}{2} = \begin{cases} \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)r + s + 1\right) \\ -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)r + s - 1\right) \end{cases} \tag{5.15}$$

$$\mathsf{h}_{r,s}^{BFK} - \frac{1}{2} = \begin{cases} \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)r + s - 1\right) \\ -\frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)r + s + 1\right) \end{cases} \tag{5.16}$$

and

$$\mathsf{h}_k^{BFK} + \frac{1}{2} = \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)k + 1\right) \tag{5.17}$$

$$\mathsf{h}_k^{BFK} - \frac{1}{2} = \frac{1}{2}\left(\left(\frac{\mathsf{c}-3}{3}\right)k - 1\right) \tag{5.18}$$

Comparing these expressions with the spectra given by (5.7), (5.8), (5.9) and (5.10) we see that the situation is the same than in the Aperiodic algebra for Verma modules built on chiral or antichiral primaries. Namely, half of the zeroes of $f_{r,s}^P = 0$ correspond to uncharged singular states and the other half correspond to charged singular states. However, since now charged and uncharged singular states share the same spectra, the zeroes of $g_k^P = 0$ can be adjudicated to either of them equivalently.

Nevertheless, if we make the choice that adding $\frac{1}{2}$ (or $-\frac{1}{2}$) to the BFK spectra (5.13) and (5.14) one gets the spectra corresponding to the chirality $+$ (or $-$) singular states, one finds the following identifications. The upper solution of $\mathsf{h}_{r,s}^{BFK} + \frac{1}{2}$ corresponds to the

charged singular states $|\chi_R\rangle_l^{(1)+}$ at level $\frac{r(s+2)}{2}$, whereas the lower solution corresponds to the uncharged singular states $|\chi_R\rangle_l^{(0)+}$ at level $\frac{rs}{2}$. The upper solution of $\mathsf{h}_{r,s}^{BFK} - \frac{1}{2}$ corresponds to the uncharged singular states $|\chi_R\rangle_l^{(0)-}$ at level $\frac{rs}{2}$, while the lower solution corresponds to the charged singular states $|\chi_R\rangle_l^{(-1)-}$ at level $\frac{r(s+2)}{2}$. The solutions $\mathsf{h}_k^{BFK}+\frac{1}{2}$ and $\mathsf{h}_k^{BFK} - \frac{1}{2}$ correspond to the charged singular states $|\chi_R\rangle_l^{(1)+}$ at level k, and $|\chi_R\rangle_l^{(-1)-}$ at level $(-k)$, respectively. Therefore, as happens with the Aperiodic NS algebra, the zeroes of the vanishing plane $g_k^P = 0$ give the solutions $\mathsf{h}_{r,s}^{(1)+}$ and $\mathsf{h}_{r,s}^{(-1)-}$ for $s = 2$, while half of the zeroes of $f_{r,s}^P = 0$ give the solutions for $s > 2$.

6 Conclusions and Final Remarks

We have shown that the determinant formulae for the N=2 Superconformal algebra can be applied directly to incomplete Verma modules, built on chiral primary states or on Ramond ground states, provided one modifies appropiately the interpretation of the zeroes in terms of identification of the levels and relative charges of the singular states.

We have written down the spectra of U(1) charges and conformal weights for the singular states of the Aperiodic NS algebra built on chiral and antichiral primary states and for the singular states of the Periodic R algebra built on the Ramond ground states, showing that, for those cases, the spectra corresponding to the charged singular states are given by two-parameter expressions, like the spectra corresponding to the uncharged singular states. This is due to the fact that only half of the zeroes of the quadratic vanishing surfaces $f_{r,s}^A = 0$ and $f_{r,s}^P = 0$ correspond to uncharged singular states, while the other half of the zeroes correspond to charged singular states, in contrast with the case of complete Verma modules for which all the zeroes of $f_{r,s}^A = 0$ and $f_{r,s}^P = 0$ correspond to uncharged singular states.

We have obtained these results simply by imposing the symmetries, dictated by the spectral flows, between charged and uncharged singular states, starting with the ansatz that the zeroes of the “chiral” determinant formulae coincide with the zeroes of the determinant formula of the Aperiodic NS algebra specialized to the cases $\Delta = \pm\frac{\mathsf{h}}{2}$.

Our results agree with all the known data (eight levels: from level $\frac{1}{2}$ until level 4). This, together with the simplicity of our derivation, strongly suggests that these results must be valid at any level.

It is already remarkable the fact that the zeroes of the determinant formulae specialized to the cases $\Delta = \pm\frac{\mathsf{h}}{2}$ (or $\Delta = \frac{\mathsf{c}}{24}$), coincide with the zeroes of the specific determinant formulae for the chiral representations of the Aperiodic algebra (or the representations of the Periodic algebra based on the Ramond ground states). The reason is that the highest weight equations resulting in the singular states, and therefore in the zeroes of

the determinant formulae, are different when one does or does not impose chirality on the primary states on which the singular states are built (or one does or does not impose that the Ramond primaries are annihilated by both G_0^+ and G_0^-). It seems that there exists a "null vector conservation law" or "null vector transmutation" when switching on and off chirality on the primary states, with charged and uncharged singular states replacing each other.

We have also analyzed in very much detail the relation between the singular states of the Twisted Topological algebra and the singular states of the Aperiodic NS algebra, showing the direct relation between their corresponding spectra. As a result we have also derived the spectrum of U(1) charges for the singular states of the Twisted Topological Algebra, which agrees with the known data (until level 4).

In addition, we have shown that the charged NS singular states built on chiral primaries have always relative charge (-1), while those built on antichiral primaries have always relative charge $(+1)$. These states are mirrored to each other under the interchange $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$ because they are the two possible untwistings of the same topological singular state of type $|\chi\rangle^{(1)G}$. In the same way, the charged R singular states built on the Ramond ground states come only in two types: $|\chi_R\rangle^{(1)+}$ and $|\chi_R\rangle^{(-1)-}$, and they are also mirrored under $H_m \leftrightarrow -H_m$, $G_r^+ \leftrightarrow G_r^-$.

In the Appendix we have analyzed thoroughly the NS singular states $|\chi_{NS}\rangle_1^{(0)}$, $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$ and $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$. We have written down the h.w. equations, with their solutions, for the cases of the primary states being non-chiral, chiral and antichiral, showing the "null vector conservation law" (or null vector transmutation) when switching on and off chirality on the primary states.

Finally, we have uncovered the existence of non-highest weight singular states, which are not secondary of any h.w. singular state.

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Appendix

The general form of the charge (+1) NS singular state at level $\frac{3}{2}$ is

$$|\chi_{NS}\rangle^{(1)}_{\frac{3}{2}} = (\alpha L_{-1}G^+_{-1/2} + \beta H_{-1}G^+_{-1/2} + \gamma G^+_{-3/2})|\Delta, \mathsf{h}\rangle\,. \tag{A.1}$$

The highest weight (h.w.) conditions $L_{m>0}|\chi\rangle = H_{m>0}|\chi\rangle = G^+_{r\geq\frac{1}{2}}|\chi\rangle = G^-_{r\geq\frac{1}{2}}|\chi\rangle = 0$, which determine the coefficients α, β, γ, as well as the conformal weight Δ and the U(1) charge h of the primary state, result as follows. For $|\Delta, \mathsf{h}\rangle$ non-chiral one obtains the equations

$$\begin{aligned}
\alpha(1+2\Delta) + \beta(1+\mathsf{h}) + 2\gamma &= 0\\
\alpha(1+\mathsf{h}) + \beta\frac{\mathsf{c}}{3} + \gamma &= 0\\
(2\alpha+\beta)(2\Delta-\mathsf{h}) + \gamma(2\Delta - 3\mathsf{h} + 2\frac{\mathsf{c}}{3}) &= 0\\
\beta(2\Delta-\mathsf{h}) - 2\gamma &= 0\\
\alpha(2\Delta-\mathsf{h}) + 2\gamma &= 0\\
\alpha+\beta &= 0\,.
\end{aligned} \tag{A.2}$$

We see that, for $\Delta \neq \frac{\mathsf{h}}{2}$, γ must necessarily be different from zero (if $\gamma = 0$ the whole vector vanishes). Hence we can choose $\gamma = 1$. Solving for the other coefficients one obtains the solution, for $\Delta \neq \frac{\mathsf{h}}{2}$

$$\alpha = \frac{-2}{2\Delta - \mathsf{h}}\ , \qquad \beta = \frac{2}{2\Delta-\mathsf{h}} \tag{A.3}$$

with $\Delta - \frac{3}{2}\mathsf{h} + \frac{\mathsf{c}-3}{3} = 0$. This solution is given by the vanishing plane $g_{3/2} = 0$, as one can check in eq. (4.2).

For the case $\Delta = \frac{\mathsf{h}}{2}$ the solution is $\gamma = 0$, $\beta = -\alpha$ and $\mathsf{h} = \frac{\mathsf{c}-3}{3}$. It is also given by the vanishing plane $g_{3/2} = 0$. If we now specialize the general solution (A.3) to the case $\Delta = -\frac{\mathsf{h}}{2}$ we find

$$\alpha = \frac{6}{\mathsf{c}-3}\ , \qquad \beta = \frac{6}{3-\mathsf{c}}\ , \qquad \mathsf{h} = \frac{\mathsf{c}-3}{6}\ . \tag{A.4}$$

For $|\Delta, \mathsf{h}\rangle$ chiral (*i.e.* $G^+_{-\frac{1}{2}}|\Delta, \mathsf{h}\rangle = 0$, $\Delta = \frac{\mathsf{h}}{2}$) the h.w. conditions only give the equation $\gamma = 0$. Therefore $|\chi_{NS}\rangle^{(1)ch}_{\frac{3}{2}}$ vanishes while $|\chi_{NS}\rangle^{(1)}_{\frac{3}{2}}$, for $\Delta = \frac{\mathsf{h}}{2}$, is a singular state with $\gamma = 0$, $\beta = -\alpha$, as we have just shown. This is a particular case of the general

result that the charged singular states built on chiral primaries have always relative charge $q=-1$, whereas those built on antichiral primaries have always $q=1$.

For $|\Delta,\mathsf{h}\rangle$ antichiral (i.e. $G^-_{-\frac{1}{2}}|\Delta,\mathsf{h}\rangle=0,\ \Delta=-\frac{\mathsf{h}}{2}$) one gets the equations

$$\begin{aligned}
\alpha(1-\mathsf{h})+\beta(1+\mathsf{h})+2\gamma &= 0\\
\alpha(1+\mathsf{h})+\beta\frac{\mathsf{c}}{3}+\gamma &= 0\\
(2\alpha+\beta)\mathsf{h}+\gamma(2\mathsf{h}-\frac{\mathsf{c}}{3}) &= 0\\
\beta\mathsf{h}+\gamma &= 0\\
\alpha(1-\mathsf{h})+\beta+\gamma &= 0\,.
\end{aligned}\tag{A.5}$$

Comparing these with equations (A.2), setting $\Delta=-\frac{\mathsf{h}}{2}$, we see that here there is one equation less and the first four equations coincide. The last equation here and the two last equations in (A.2) are different. These equations correspond to the h.w. condition $G^-_{\frac{1}{2}}|\chi_{NS}\rangle^{(1)}_{\frac{3}{2}}=0$. As before $\gamma\neq0$ necessarily, thus we set $\gamma=1$. Solving for the other coefficients and for h one obtains two solutions:

$$\alpha=\begin{cases}\frac{6}{\mathsf{c}-3}\\ \frac{\mathsf{c}-3}{6}\end{cases},\qquad \beta=\begin{cases}\frac{6}{3-\mathsf{c}}\\ -1\end{cases},\qquad \mathsf{h}=\begin{cases}\frac{\mathsf{c}-3}{6}\\ 1\end{cases}.\tag{A.6}$$

The solution $\mathsf{h}=\frac{\mathsf{c}-3}{6}$ is the solution given by the vanishing plane $g_{\frac{3}{2}}=0$, and therefore the only solution for $\Delta=-\frac{\mathsf{h}}{2}$, $|\Delta,\mathsf{h}\rangle$ non-chiral, as we have shown in eq. (A.4). The solution $\mathsf{h}=1$ corresponds to $\hat{\mathsf{h}}_{1,2}$ in eq.(4.4), given by the vanishing quadratic surface $f_{1,2}=0$. These two solutions are given, on the other hand, by $\mathsf{h}^{(1)}_{2,2}$ and $\mathsf{h}^{(1)}_{1,4}$ in eq. (4.10).

Similarly, the general form of the charge (-1) NS singular state at level $\frac{3}{2}$ is

$$|\chi_{NS}\rangle^{(-1)}_{\frac{3}{2}}=(\alpha L_{-1}G^-_{-1/2}+\beta H_{-1}G^-_{-1/2}+\gamma G^-_{-3/2})|\Delta,\mathsf{h}\rangle\,.\tag{A.7}$$

Since this case is very similar to the previous one we will consider only the chiral representations. The h.w. conditions result in $\gamma=0$ for $|\Delta,\mathsf{h}\rangle$ antichiral (therefore $|\chi_{NS}\rangle^{(-1)a}_{\frac{3}{2}}$ vanishes), while for $|\Delta,\mathsf{h}\rangle$ chiral one gets the equations

$$\begin{aligned}
\alpha(1+\mathsf{h})+\beta(\mathsf{h}-1)+2\gamma &= 0\\
\alpha(\mathsf{h}-1)+\beta\frac{\mathsf{c}}{3}-\gamma &= 0
\end{aligned}$$

$$
\begin{aligned}
(2\alpha-\beta)\mathsf{h}+\gamma(2\mathsf{h}+\frac{\mathsf{c}}{3}) &= 0\\
\beta\mathsf{h}+\gamma &= 0\\
\alpha(\mathsf{h}+1)-\beta+\gamma &= 0\,,
\end{aligned}
\tag{A.8}
$$

where, as before, we can set $\gamma=1$. The other coefficients and the U(1) charge h read

$$
\alpha=\left\{\begin{array}{l}\frac{6}{\mathsf{c}-3}\\ \frac{\mathsf{c}-3}{6}\end{array}\right.,\qquad \beta=\left\{\begin{array}{l}\frac{6}{\mathsf{c}-3}\\ 1\end{array}\right.,\qquad \mathsf{h}=\left\{\begin{array}{l}\frac{3-\mathsf{c}}{6}\\ -1\end{array}\right. . \tag{A.9}
$$

These solutions correspond to $(-\mathsf{h}^{(1)}_{2,2})$ and $(-\mathsf{h}^{(1)}_{1,4})$ respectively (the upper solution is the solution given by the vanishing plane $g_{-\frac{3}{2}}$). We see that $|\chi_{NS}\rangle^{(1)a}_{\frac{3}{2}}$ and $|\chi_{NS}\rangle^{(-1)ch}_{\frac{3}{2}}$ are mirrored under the interchange $H_m\leftrightarrow -H_m$, $G^+_r\leftrightarrow G^-_r$, reflecting the fact that they are the two different untwistings of the same topological singular state $|\chi\rangle^{(1)G}_2$, given by

$$
|\chi\rangle^{(1)G}_2=(\mathcal{G}_{-2}+\alpha\mathcal{L}_{-1}\mathcal{G}_{-1}+\beta\mathcal{H}_{-1}\mathcal{G}_{-1})|\phi\rangle_{\mathsf{h}} \tag{A.10}
$$

with

$$
\alpha=\left\{\begin{array}{l}\frac{6}{\mathsf{c}-3}\\ \frac{\mathsf{c}-3}{6}\end{array}\right.,\qquad \beta=\left\{\begin{array}{l}\frac{6}{3-\mathsf{c}}\\ -1\end{array}\right.,\qquad \mathsf{h}=\left\{\begin{array}{l}\frac{\mathsf{c}-3}{6}\\ 1\end{array}\right. . \tag{A.11}
$$

as the reader can verify using the twists (2.2) and (2.3).

The general form of the uncharged NS singular state at level 1 is given by

$$
|\chi_{NS}\rangle^{(0)}_1=(\alpha L_{-1}+\beta H_{-1}+\gamma G^+_{-1/2}G^-_{-1/2})|\Delta,\mathsf{h}\rangle\,. \tag{A.12}
$$

The h.w. conditions result as follows. For $|\Delta,\mathsf{h}\rangle$ non-chiral one obtains the equations

$$
\begin{aligned}
\alpha\ 2\Delta+\beta\ \mathsf{h}+\gamma(2\Delta+\mathsf{h}) &= 0\\
\alpha\ \mathsf{h}+\beta\ \frac{\mathsf{c}}{3}+\gamma(2\Delta+\mathsf{h}) &= 0\\
\alpha-\beta-\gamma(2\Delta+\mathsf{h}) &= 0\\
\alpha+\beta+\gamma(2\Delta-\mathsf{h}+2) &= 0\,.
\end{aligned}
\tag{A.13}
$$

As before we can set $\gamma=1$ and we get

$$
\alpha=\mathsf{h}-1,\qquad \beta=-(2\Delta+1)\,, \tag{A.14}
$$

with $\mathsf{h}^2 - 2\Delta\frac{\mathsf{c}-3}{3} - \frac{\mathsf{c}}{3} = 0$. This solution corresponds to the quadratic vanishing surface $f_{12} = 0$ in (4.1). It was given before in [10] and [11].

For $\Delta = \frac{\mathsf{h}}{2}$ the solutions are

$$\alpha = \begin{cases} \frac{\mathsf{c}-3}{3} \\ -2 \end{cases}, \quad \hat{\alpha} = \begin{cases} \frac{\mathsf{c}+3}{3} \\ 0 \end{cases}, \quad \beta = \begin{cases} -\frac{\mathsf{c}+3}{3} \\ 0 \end{cases}, \quad \mathsf{h} = \begin{cases} \frac{\mathsf{c}}{3} \\ -1 \end{cases}, \tag{A.15}$$

α transforming into $\hat{\alpha}$ if we commute the term $G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \to G^-_{-\frac{1}{2}}G^+_{-\frac{1}{2}}$. These solutions correspond to $(-\mathsf{h}_{1,2})$ and $(-\hat{\mathsf{h}}_{1,2})$ in (4.3) and (4.4) respectively.

For $\Delta = -\frac{\mathsf{h}}{2}$ the solutions are

$$\alpha = \begin{cases} -\frac{\mathsf{c}+3}{3} \\ 0 \end{cases}, \quad \beta = \begin{cases} -\frac{\mathsf{c}+3}{3} \\ 0 \end{cases}, \quad \mathsf{h} = \begin{cases} -\frac{\mathsf{c}}{3} \\ 1 \end{cases}. \tag{A.16}$$

These solutions correspond to $\mathsf{h}_{1,2}$ and $\hat{\mathsf{h}}_{1,2}$ in eqns. (4.3) and (4.4) respectively.

For the case of $|\Delta, \mathsf{h}\rangle$ being chiral we can set $\gamma = 0$ and the h.w. conditions on $|\chi_{NS}\rangle_1^{(0)ch}$ give the equations

$$\begin{aligned} \alpha + \beta &= 0 \\ \alpha\, \mathsf{h} + \beta\, \frac{\mathsf{c}}{3} &= 0\,. \end{aligned} \tag{A.17}$$

Comparing these with eqns. (A.13) for $\Delta = \frac{\mathsf{h}}{2}$, we see that the first and fourth equations in (A.13) coincide now with the first one here, while the third equation in (A.13), which corresponds to the h.w. condition $G^+_{\frac{1}{2}}|\chi\rangle = 0$, disappears, the reason being that the complete equation reads $(\alpha - \beta - \gamma(2\Delta + \mathsf{h}))G^+_{-\frac{1}{2}}|\Delta, \mathsf{h}\rangle = 0$. The solution to these equations is $\beta = -\alpha, \quad \mathsf{h} = \frac{\mathsf{c}}{3}$, that is

$$|\chi_{NS}\rangle_1^{(0)ch} = (L_{-1} - H_{-1})\ \ |\Delta = \frac{\mathsf{c}}{6},\ \mathsf{h} = \frac{\mathsf{c}}{3}\rangle\,. \tag{A.18}$$

Therefore, only the solution $(-\mathsf{h}_{1,2}) = \frac{\mathsf{c}}{3}$ remains after switching on chirality on the primary state $|\Delta, \mathsf{h}\rangle$, while $(-\hat{\mathsf{h}}_{1,2}) = -1$ is not a solution anymore. Not only that, but in addition the singular state $|\chi_{NS}\rangle_1^{(0)}$ vanishes for $(-\hat{\mathsf{h}}_{1,2})$ when one imposes chirality on $|\Delta, \mathsf{h}\rangle$, as one can check in (A.15).

For the case of $|\Delta, \mathsf{h}\rangle$ being antichiral the h.w. conditions on $|\chi_{NS}\rangle_1^{(0)a}$ give the equations

$$
\begin{aligned}
\alpha - \beta &= 0 \\
\alpha\, \mathsf{h} + \beta\, \frac{\mathsf{c}}{3} &= 0\,.
\end{aligned} \tag{A.19}
$$

Comparing these with eqns. (A.13) we see that now the last equation in (A.13), which corresponds to the h.w. condition $G^-_{\frac{1}{2}}|\chi\rangle = 0$, has disappeared, the reason being that the complete equation reads $(\alpha+\beta+\gamma(2\Delta-\mathsf{h}+2))G^-_{-\frac{1}{2}}|\Delta,\mathsf{h}\rangle = 0$. The solution to these equations is $\beta=\alpha, \quad \mathsf{h}=-\frac{\mathsf{c}}{3}$, that is

$$
|\chi_{NS}\rangle_1^{(0)a} = (L_{-1}+H_{-1})\ \ |\Delta=\frac{\mathsf{c}}{6},\ \mathsf{h}=-\frac{\mathsf{c}}{3}\rangle\,. \tag{A.20}
$$

Therefore, only the solution $\mathsf{h}_{1,2}=-\frac{\mathsf{c}}{3}$ remains after switching on antichirality on the primary state $|\Delta,\mathsf{h}\rangle$, while the solution $\hat{\mathsf{h}}_{1,2}=1$ disappears. In addition, as happened in the chiral case, the singular state $|\chi_{NS}\rangle_1^{(0)}$ vanishes for $\hat{\mathsf{h}}_{1,2}$ when one imposes antichirality on $|\Delta,\mathsf{h}\rangle$, as one can check in (A.16).

Observe that $|\chi_{NS}\rangle_1^{(0)ch}$ and $|\chi_{NS}\rangle_1^{(0)a}$ are symmetric under the interchange $H_m \leftrightarrow -H_m,\ G^+_r \leftrightarrow G^-_r$, as expected. This also happens for the singular states $|\chi_{NS}\rangle_1^{(0)}$ specialized to the cases $\Delta=\frac{\mathsf{h}}{2}$ and $\Delta=-\frac{\mathsf{h}}{2}$, eqns. (A.16) and (A.15), as one can check (one has to take into account that the commutation $G^+_{-1/2}G^-_{-1/2}\to G^-_{-1/2}G^+_{-1/2}$ produces a global minus sign).

We have seen that the uncharged states $|\chi_{NS}\rangle_1^{(0)}$, for $\mathsf{h}=\hat{\mathsf{h}}_{1,2}=1$ $(\mathsf{h}=-\hat{\mathsf{h}}_{1,2}=-1\ )$, $\Delta=-\frac{1}{2}$, vanish when one switches on antichirality (chirality) respectively on $|\Delta,\mathsf{h}\rangle$. An interesting related question now is what happens with the charged singular states $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ for $\mathsf{h}=\hat{\mathsf{h}}_{1,2}=1$ and $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$ for $\mathsf{h}=-\hat{\mathsf{h}}_{1,2}=-1$ when one switches off antichirality or chirality on $|\Delta,\mathsf{h}\rangle$; that is, which kind of states these singular states become once they are not h.w. singular states anymore. One might think that these h.w. charged singular states which disappear at level $\frac{3}{2}$ turn into secondary singular states of the h.w. uncharged singular states which appear at level 1. However this is not quite true because, as we will see, these charged states, which are still singular although not h.w., can descend down to the h.w. uncharged singular states, but not the other way around.

To see this in some detail let us write the h.w. uncharged singular states at level 1 for $\mathsf{h}=\pm1$, $\Delta=\mp\frac{\mathsf{h}}{2}=-\frac{1}{2}$, given by eqns. (A.16) and (A.15). For $\mathsf{h}=1,\ \Delta=-\frac{1}{2}$, the singular state is

$$
|\chi_{NS}\rangle_1^{(0)} = G^+_{-1/2}G^-_{-1/2}\ \ |\Delta=-\frac{1}{2},\ \mathsf{h}=1\rangle\,. \tag{A.21}
$$

Its level $\frac{3}{2}$ descendant with relative charge $q = 1$ vanishes since $G^+_{-1/2}\ |\chi_{NS}\rangle^{(0)}_1 = 0$. Similarly, for $\mathsf{h} = -1,\ \Delta = -\frac{1}{2}$, the singular state is

$$|\chi_{NS}\rangle^{(0)}_1 = G^-_{-1/2}G^+_{-1/2}\ \ |\Delta = -\frac{1}{2},\ \mathsf{h} = -1\rangle\,, \tag{A.22}$$

and its level $\frac{3}{2}$ descendant with relative charge $q = -1$ vanishes. (It is also straightforward to see that the uncharged singular states (A.21) and (A.22) vanish when one imposes antichirality and chirality, respectively, on the primary states, as we said before.)

On the other hand, the h.w. charged singular states for $\mathsf{h} = \pm 1$, $\Delta = \mp\frac{\mathsf{h}}{2} = -\frac{1}{2}$, given by (A.6) and (A.9), are

$$|\chi_{NS}\rangle^{(1)a}_{\frac{3}{2}} = (\frac{\mathsf{c}-3}{6}\ L_{-1}G^+_{-1/2} - H_{-1}G^+_{-1/2}\ +\ G^+_{-3/2})\ \ |\Delta = -\frac{1}{2},\ \mathsf{h} = 1\rangle \tag{A.23}$$

and

$$|\chi_{NS}\rangle^{(-1)ch}_{\frac{3}{2}} = (\frac{\mathsf{c}-3}{6}\ L_{-1}G^-_{-1/2} + H_{-1}G^-_{-1/2}\ +\ G^-_{-3/2})\ \ |\Delta = -\frac{1}{2},\ \mathsf{h} = -1\rangle \tag{A.24}$$

If we now switch off antichirality (chirality) on the primary state $|\Delta, \mathsf{h}\rangle$, then the h.w. condition $G^-_{1/2}|\chi_{NS}\rangle^{(1)a}_{\frac{3}{2}} = 0$ $\ (G^+_{1/2}|\chi_{NS}\rangle^{(-1)ch}_{\frac{3}{2}} = 0\ )$ is not satisfied anymore, although the states are still singular, *i.e.* have zero norm, as the reader can verify. However, these states cannot be secondary states of the uncharged singular states (A.21) and (A.22), as we have just discussed, nor are they secondary states of any level $\frac{1}{2}$ singular states. In other words, $|\chi_{NS}\rangle^{(1)a}_{\frac{3}{2}}$ and $|\chi_{NS}\rangle^{(-1)ch}_{\frac{3}{2}}$ become non-highest weight singular states, not secondary of any singular states, once we switch off antichirality and chirality on the primary state $|\Delta, \mathsf{h}\rangle$, respectively.

Nevertheless, these level $\frac{3}{2}$ zero-norm states do descend to the level 1 uncharged singular states (A.21) and (A.22) under the action of $G^-_{1/2}$ and $G^+_{1/2}$, respectively, as is easy to check, but not the other way around. The reason is that the singular states (A.21) and (A.22) do not build complete Verma modules, as is the usual case for the h.w. singular states of the N=2 superconformal algebra.

To finish let us discuss further the mechanism of "null vector conservation" or "transmutation". For this it is instructive to analyze the behaviour of the uncharged singular state $|\chi_{NS}\rangle^{(0)}_1$ and its level $\frac{3}{2}$ descendants near the limits $\mathsf{h} \to \pm 1,\ \Delta \to -\frac{1}{2}$.

Let us start with h near 1. Thus we set $\mathsf{h} = 1 + \epsilon$, $\Delta = -\frac{1}{2}(1+\delta)$, Δ and h satisfying the quadratic vanishing surface relation, which results in

$$\frac{\mathsf{c}}{3} = \frac{\delta - 2\epsilon - \epsilon^2}{\delta} \,. \tag{A.25}$$

The state $|\chi_{NS}\rangle_1^{(0)}$ is expressed now as

$$|\chi_{NS}\rangle_1^{(0)} = (\epsilon L_{-1} + \delta H_{-1} + G^+_{-1/2} G^-_{-1/2}) |\Delta, \mathsf{h}\rangle \,. \tag{A.26}$$

Its charge (+1) descendant at level $\frac{3}{2}$, which we denote as $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$, is a singular state which, in principle, is not h.w.

$$|\Upsilon\rangle_{\frac{3}{2}}^{(1)} = G^+_{-1/2} |\chi_{NS}\rangle_1^{(0)} = (\epsilon L_{-1} G^+_{-1/2} + \delta H_{-1} G^+_{-1/2} - \delta G^+_{-3/2}) |\Delta, \mathsf{h}\rangle \,.$$

Now comes a subtle point. In principle we can normalize $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$ in the same way as $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$, *i.e.* dividing all the coefficients by $(-\delta)$ so that the coefficient of $G^+_{-3/2}$ is 1, resulting in

$$|\Upsilon\rangle_{\frac{3}{2}}^{(1)} = ((-\frac{\epsilon}{\delta}) L_{-1} G^+_{-1/2} - H_{-1} G^+_{-1/2} + G^+_{-3/2}) |\Delta, \mathsf{h}\rangle \,. \tag{A.27}$$

However, these two normalizations are not equivalent when taking the limit $(\mathsf{h} = 1, \Delta = -\frac{1}{2})$,*i.e.* $(\epsilon \to 0, \delta \to 0)$. Namely $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$ vanishes with the first normalization whereas with the second normalization it becomes the h.w. singular state $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$, since $(-\frac{\epsilon}{\delta})$ turns into $(\frac{\mathsf{c}-3}{6})$, as can be deduced easily from (A.25).

It seems that the two normalizations distinguish whether the primary state $|\Delta, \mathsf{h}\rangle$ approaches an antichiral or a non-chiral state as $\mathsf{h} \to 1, \Delta \to -\frac{1}{2}$. This is indeed true, the reason is that if we normalize $|\chi_{NS}\rangle_1^{(0)}$ in the same way as its descendant $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$, then in the second normalization, *i.e.* dividing by $(-\delta)$, it blows up approaching the limit $(\epsilon \to 0, \delta \to 0)$ unless the term $G^+_{-1/2} G^-_{-1/2}$ goes away, exactly what happens if $|\Delta, \mathsf{h}\rangle$ is antichiral. But the resulting uncharged state without the term $G^+_{-1/2} G^-_{-1/2}$ is not a singular state anymore. The action of $G^+_{-1/2}$ on this state results precisely in the charged h.w. singular state $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$.

We see therefore that, in the limit $(\mathsf{h} = 1, \Delta = -\frac{1}{2})$, the first normalization produces the h.w. uncharged singular state at level 1 $|\chi_{NS}\rangle_1^{(0)}$, with a vanishing charge (+1) descendant at level $\frac{3}{2}$, whereas the second normalization produces the h.w. charge (+1) singular state at level $\frac{3}{2}$ $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$.

Repeating this analysis for the case h near -1 we find that in the limit $(\mathsf{h} = -1, \Delta = -\frac{1}{2})$, the first normalization produces the h.w. uncharged singular state at level 1 $|\chi_{NS}\rangle_1^{(0)}$, with a vanishing charge (-1) descendant at level $\frac{3}{2}$, whereas the second normalization produces the h.w. charge (-1) singular state at level $\frac{3}{2}$ $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$.

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